INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2009; **61**:987–994
Published online 15 January 2009 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.1994

On comparison of exact and series solutions for thin film flow of a third-grade fluid

S. Abbasbandy[1,*,†], T. Hayat[2], F. M. Mahomed[3] and R. Ellahi[4]

[1]*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran 14778, Iran*
[2]*Department of Mathematics, Quaid-i-Azam University, Islamabad 44000, Pakistan*
[3]*School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa*
[4]*Department of Mathematics, Faculty of Applied Sciences, IIU, Islamabad, Pakistan*

SUMMARY

The present investigation derives the exact and series solutions for steady thin film flow of a third-grade fluid. The series solution is constructed by a homotopy analysis method. The obtained solutions are compared and an excellent agreement between these is achieved. Copyright © 2009 John Wiley & Sons, Ltd.

Received 15 September 2008; Revised 13 November 2008; Accepted 3 December 2008

KEY WORDS: thin film flow; third-grade fluid; analytic solution; homotopy analysis method (HAM); serious solution; exact solution

1. INTRODUCTION

Over the last four decades, the analytic solutions to flows of non-Newtonian fluids have attracted the attention of various researchers [1–10]. The literature dealing with such solutions is scarce. This is in fact due to fairly complicated constitutive equations of non-Newtonian fluids. These equations involve a number of complex parameters that in general add more number of terms and increase the order of the governing equations. In spite of all these challenges some researchers in the field [11–20] are recently engaged in developing the analytic solutions. Very recently, Hayat *et al.* [21] provide exact solutions for thin film flow of a third-grade fluid down an inclined plane.

*Correspondence to: S. Abbasbandy, Department of Mathematics, Imam Khomeini International University, Ghazvin 34149-16818, Iran.
†E-mail: abbasbandy@yahoo.com

Contract/grant sponsor: Higher Education Commission (HEC) of Pakistan

The purpose of the present communication is to provide the series solution of the problem considered in [21]. The series solution is given by a homotopy analysis method (HAM) developed by Liao [22, 23]. Convergence of the obtained series solution is explicitly discussed. In addition, an exact solution by incorporating typo mistake is included. Finally, the numerical values are tabulated in order to show the comparison between the exact and homotopy solutions. Like several previous attempts [24–34], the constructed homotopy solution here yields reliable results.

2. PROBLEM STATEMENT AND EXACT SOLUTION

In this section, we reconsider the thin film flow problem in a third-grade fluid down an incline plane of inclination $\alpha \neq 0$ for the homotopy solution [21]. The surface tension is neglected and ambient air is taken stationary. Thickness of the thin film is kept uniform. Neglecting the pressure gradient, dimensionless problem through conservations of mass and momentum is in the form [21]

$$\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+6\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+1=0 \tag{1}$$

$$u(0)=0, \quad u'(1)+2\beta(u'(1))^3=0 \tag{2}$$

where

$$y=\delta y^*, \quad u=\frac{\delta^2\rho g\sin(\alpha)}{\mu}u^*, \quad \beta^*=\frac{3\delta^2\rho^2g^2\sin^2(\alpha)}{\mu^3}\beta$$

and asterisks have been dropped for simplicity. In the above expressions, μ is the dynamic viscosity, g is the gravity, ρ is the density and $\beta\ (=\beta_2+\beta_3)>0$, $\beta_i\ (i=2,3)$ is the material constants. It is pointed out that an exact solution of the problem consisting of Equations (1) and (2) is obtained in the study [21]. However, the presented exact solution has an error due to typo graphical mistake. The correct form of the exact solution is

$$u_{\mathrm{Exact}}(y)=\frac{9(-y+1)}{8(6)^{2/3}}\left(\sqrt[3]{\frac{\sqrt{81(-y+1)^2+\frac{6}{\beta}}-9(-y+1)}{\beta}}-\sqrt[3]{\frac{\sqrt{81(-y+1)^2+\frac{6}{\beta}}+9(-y+1)}{\beta}}\right)$$

$$+\frac{1}{24(6)^{2/3}}\sqrt{81(-y+1)^2+\frac{6}{\beta}}\left(\sqrt[3]{\frac{\sqrt{81(-y+1)^2+\frac{6}{\beta}}+9(-y+1)}{\beta}}\right.$$

DOI: 10.1002/fld

$$+\sqrt[3]{\frac{\sqrt{81(-y+1)^2+\frac{6}{\beta}}-9(-y+1)}{\beta}}\Bigg)+\frac{9}{8(6)^{2/3}}\left(\sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}+9}{\beta}}-\sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}-9}{\beta}}\right)$$

$$-\frac{1}{24(6)^{2/3}}\sqrt{81+\frac{6}{\beta}}\left(\sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}+9}{\beta}}+\sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}-9}{\beta}}\right) \tag{3}$$

In the following section, we develop the series solution of the problem.

3. SOLUTION BY HAM

By Equation (1) and the boundary conditions (2), solution can be expressed in the form

$$u(y)=\sum_{m=1}^{+\infty} a_m y^m \tag{4}$$

where a_m is a coefficient to be determined later. According to the *rule of solution expression* denoted by Equation (4) and the boundary conditions (2), the quadratic approximation

$$u_0(y)=\varepsilon\left(-y+\frac{y^2}{2}\right) \tag{5}$$

can be chosen as the initial approximation to $u(y)$, where $\varepsilon<0$ is a coefficient to be determined later. An auxiliary linear operator $\mathscr{L}$ is defined by

$$\mathscr{L}[\phi(y;p)]=\left(\frac{\partial^2}{\partial y^2}\right)\phi(y;p) \tag{6}$$

which satisfy

$$\mathscr{L}[C_1+C_2 y]=0 \tag{7}$$

in which C_1 and C_2 are constants. Considering Equation (1), the nonlinear operator is given by

$$\mathscr{N}[\phi(y;p)]=\frac{\partial^2\phi}{\partial y^2}+6\beta\left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2}+1 \tag{8}$$

and the homotopy is constructed as follows:

$$\mathscr{H}[\phi(y;p)]=(1-p)\mathscr{L}[\phi(y;p)-u_0(y)]-\hbar p\mathscr{N}[\phi(y;p)] \tag{9}$$

DOI: 10.1002/fld

where $\hbar \neq 0$ is the convergence-control parameter [22, 23]. Setting $\mathscr{H}[\phi(y;p)]=0$, the zero-order deformation problem is

$$
\begin{gathered}
(1-p)\mathscr{L}[\phi(y;p)-u_0(y)]=\hbar p\mathscr{N}[\phi(y;p)] \\
\phi(0;p)=0, \quad (1-p)\left[\frac{\partial\phi(y;p)}{\partial y}\right]_{y=1}+\hbar p\left[\frac{\partial\phi(y;p)}{\partial y}+2\beta\left(\frac{\partial\phi(y;p)}{\partial y}\right)^3\right]_{y=1}=0
\end{gathered}
\tag{10}
$$

where $p\in[0,1]$ is an embedding parameter. When the parameter p increases from 0 to 1, the solution $\phi(y;p)$ varies from $u_0(y)$ to $u(y)$. If this continuous variation is smooth enough, the Maclaurin's series with respect to p can be constructed for $\phi(y;p)$, and further, if this series is convergent at $p=1$, we have

$$
\begin{aligned}
u(y) &= u_0(y)+\sum_{m=1}^{+\infty} u_m(y)=\sum_{m=0}^{+\infty}\phi_m(y,\hbar) \\
u_m(y) &= \frac{1}{m!}\frac{\partial^m\phi(y;p)}{\partial p^m}\bigg|_{p=0}
\end{aligned}
\tag{11}
$$

Differentiating Equation (10) and related conditions m times with respect to p, then setting $p=0$, and finally dividing by $m!$, the mth-order deformation problem can be written as

$$
\mathscr{L}[u_m(y)-\chi_m u_{m-1}(y)]=\hbar R_m(y) \quad (m=1,2,3,\ldots) \tag{12}
$$

$$
\begin{gathered}
u_m(0)=0 \\
u'_m(1)-\chi_m u'_{m-1}(1)+\hbar\left(u'_{m-1}(1)+2\beta\sum_{n=0}^{m-1}\left(\sum_{i=0}^{n}u'_i(1)u'_{n-i}(1)\right)u'_{m-n-1}(1)\right)=0
\end{gathered}
\tag{13}
$$

where

$$
R_m(y)=u''_{m-1}+6\beta\sum_{n=0}^{m-1}\left(\sum_{i=0}^{n}u'_i u'_{n-i}\right)u''_{m-n-1}+(1-\chi_m)
$$

and prime denotes differentiation with respect to y and

$$
\chi_m=\begin{cases}0, & m\leqslant 1\\ 1, & m>1\end{cases}
$$

The general solution of Equation (12) is of the form

$$
u_m(y)=\hat{u}_m(y)+C_1+C_2 y \tag{14}
$$

where $\hat{u}_m(y)$ is particular solution of Equation (12), C_1 and C_2 are constants that will be obtained by conditions (13).

At the Nth-order approximation, the analytic solution of Equation (1) is

$$
u(y)\approx U_N(y)=\sum_{i=0}^{N}u_i(y) \tag{15}
$$

DOI: 10.1002/fld

Note that the auxiliary parameter $\hbar$ is employed to adjust the convergence region of the series (15) in the homotopy analysis solution. By $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [22], the appropriate region for $\hbar$ is a horizontal line segment.

4. NUMERICAL RESULTS

We use the widely applied symbolic computation software MATHEMATICA to solve Equation (12) and find that $\phi_m(y,\hbar)$ has the following structure:

$$\phi_n(y,\hbar)=\sum_{m=1}^{2n+2}\Phi_{n,m}(\varepsilon,\hbar)y^m, \quad n\geqslant 0$$

For example, we can find that

$$\Phi_{1,1}=-((1+\hbar)-1)-\varepsilon(1+\hbar)-10\hbar\varepsilon^3$$

$$\Phi_{2,1}=-((1+\hbar)^2-1)-\varepsilon(1+\hbar)^2-30\hbar^2\varepsilon^2-20\hbar\varepsilon^3-40\hbar^2\varepsilon^3-300\hbar^2\varepsilon^5$$

and so on. The parameter ε is used to optimize the initial approximation of $u(y)$. Here, for choosing ε we minimize the absolute residual error of Equation (1) at $y=0$ for $u_0(y)$ and each β. Table I shows the obtained values for ε for some examples.

By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$, which ensures the convergence of the solution series. As pointed out by Liao [22], the appropriate region for $\hbar$ is a horizontal line segment. We can investigate the influence of $\hbar$ on the convergence of $u'(0)$, by plotting the curve of it versus $\hbar$, as shown in Figures 1 and 2 for some examples. We can see that by decreasing β, the distance of reasonable interval for $\hbar$ increases. By considering the $\hbar$-curve we can obtain the reasonable interval for $\hbar$ in each case. In addition, by computing the error of norm 2 for two successive approximation of $U_N(y)$, we can obtain the best value for $\hbar$ for every β. Table I shows the chosen values for $\hbar$ for some examples and Figure 3 shows the HAM solutions for $U_{20}(y)$ for these values of $\hbar$.

The so-called homotopy-Padé technique (see [22]) is employed, which greatly accelerates the convergence. The $[r,s]$ homotopy-Padé approximations of $u'(0)$ and $u''(0)$, according to (11), are

Table I. Results for ε and $\hbar$.

β	$\frac{1}{3}$	$\frac{2}{3}$	1	5	10	20
ε	−0.59	−0.5	−0.451	−0.287	−0.22	−0.189
$\hbar$	−0.490	−0.399	−0.352	−0.181	−0.126	−0.097

DOI: 10.1002/fld

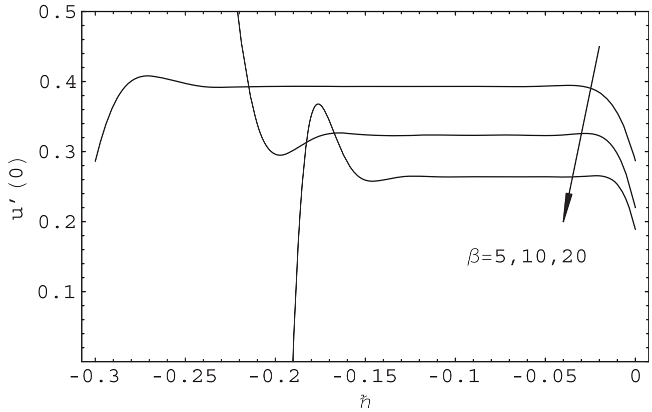

Figure 1. The $\hbar$-curves of $u'(0)$ for the 20th-order approximation.

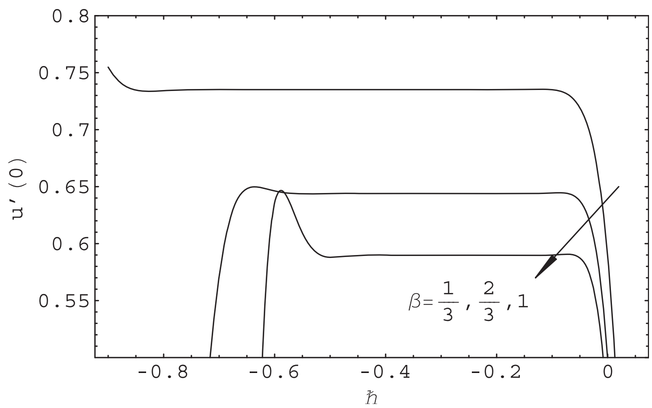

Figure 2. The $\hbar$-curves of $u'(0)$ for the 20th-order approximation.

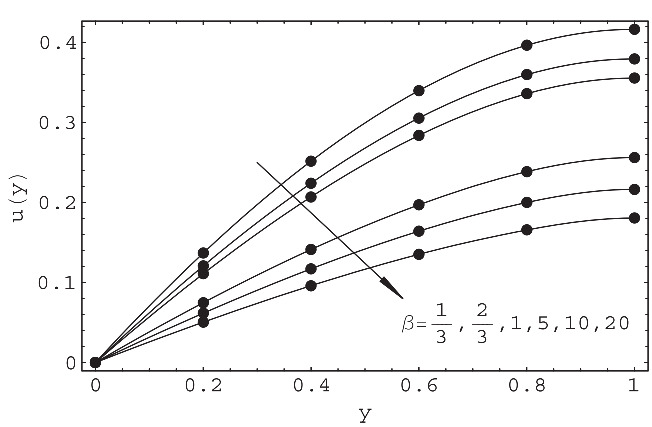

Figure 3. Dimensionless velocity profiles with different values of β for the 20th-order approximation. Symbols: exact solution from (3); solid curves: HAM solution.

DOI: 10.1002/fld

Table II. Results for [10, 10] homotopy-Padé approach for $u'(0)$ and $u''(0)$.

β	$u'(0)$	$u'_{\text{Exact}}(0)$	$u''(0)$	$u''_{\text{Exact}}(0)$
$\frac{1}{3}$	0.735139	0.735139	−0.480571	−0.480571
$\frac{2}{3}$	0.643955	0.643955	−0.376122	−0.376122
1	0.589755	0.589755	−0.323954	−0.323954
5	0.393003	0.393003	−0.177508	−0.177508
10	0.323415	0.323418	−0.137438	−0.137439
20	0.264000	0.264001	−0.106796	−0.106797

formulated by

$$\frac{\sum_{k=0}^{r}\phi'_k(0,\hbar)}{1+\sum_{k=1}^{s}\phi'_{r+k+1}(0,\hbar)}, \quad \frac{\sum_{k=0}^{r}\phi''_k(0,\hbar)}{1+\sum_{k=1}^{s}\phi''_{r+k+1}(0,\hbar)}$$

respectively. In many cases, the $[r,r]$ homotopy-Padé approximation does not depend upon the auxiliary parameter $\hbar$. Table II shows the [10, 10] homotopy-Padé results for $u'(0)$ and $u''(0)$ for some examples.

5. FINAL REMARKS

In the present study, both exact and homotopy analysis solutions are first derived and then compared. It is explicitly shown that an agreement between the derived solutions is excellent. Hence, the presented analysis tends further confidence and reliability of HAM.

ACKNOWLEDGEMENTS

We acknowledge the financial support provided by the Higher Education Commission (HEC) of Pakistan. The authors would like to thank anonymous referees for valuable suggestions.

REFERENCES

1. Rajagopal KR, Fosdick RL. Thermodynamics and stability of fluids of third grade. *Proceedings of the Royal Society of London*, *Series A* 1980; **369**:351–377.
2. Rajagopal KR. Flow of viscoelastic fluids between rotating disks. *Theoretical and Computational Fluid Dynamics* 1992; **3**:185–206.
3. Man CS. Nonsteady channel flow of ice as a modified second-order fluid with power law viscosity. *Archive for Rational Mechanics and Analysis* 1992; **119**:35–57.
4. Franchi H, Straughan B. Stability and nonexistence results in the generalized theory of a fluid of second grade. *Journal of Mathematical Analysis and Applications* 1993; **180**:122–137.
5. Dunn JE, Rajagopal KR. Fluids of differential type: critical review and thermodynamic analysis. *International Journal of Engineering Science* 1995; **33**:689–729.
6. Choi JJ, Rusak Z, Tichy JA. Maxwell fluid suction flow in a channel. *Journal of Non-Newtonian Fluid Mechanics* 1999; **85**:165–187.
7. Aksoy Y, Pakdemirli M, Khalique CM. Boundary layer equations and stretching sheet solutions for the modified second grade fluid. *International Journal of Engineering Science* 2007; **45**:829–841.

DOI: 10.1002/fld

8. Sajid M, Hayat T. Series solution for steady flow of a third grade fluid through porous space. *Transport in Porous Media* 2008; **71**:173–183.
9. Hayat T, Sajid M, Ayub M. On explicit analytic solution for MHD pipe flow of a fourth grade fluid. *Communications in Nonlinear Science and Numerical Simulation* 2008; **13**:745–751.
10. Hayat T, Abbas Z, Javed T. Mixed convection flow of a micropolar fluid over a nonlinearly stretching sheet. *Physics Letters A* 2008; **372**:637–647.
11. Fetecau C, Hayat T. Starting solutions for oscillating motions of Oldroyd-B fluids in cylindrical domains. *Journal of Non-Newtonian Fluid Mechanics* 2008; **153**:191–201.
12. Vieru D, Fetecau C. Flow of a generalized Oldroyd-B fluid due to a constantly accelerating plate. *Applied Mathematics and Computation* 2008; **201**:834–842.
13. Fetecau C, Vieru D. A note on the second problem of Stokes for Newtonian fluids. *International Journal of Non-linear Mechanics* 2008; **43**:302–319.
14. Xue CF, Nie JX, Tan WC. An exact solution of start-up flow for the fractional generalized Burgers' fluid in a porous half space. *Nonlinear Analysis* 2008; **69**:2086–2094.
15. Shen F, Tan WC, Zhao YH, Masuoka T. The Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivative model. *Nonlinear Analysis*: *Real World Applications* 2006; **7**:1072–1080.
16. Hayat T, Ellahi R, Asghar S. Hall effects on unsteady flow due to non-coaxially rotating disk and a fluid at infinity. *Chemical Engineering Communications* 2008; **195**:958–976.
17. Hayat T, Asghar S, Khalique CM, Ellahi R. Influence of a partial slip on flows of a second grade fluid in a porous medium. *Journal of Porous Media* 2007; **10**:797–805.
18. Asghar S, Hayat T, Kara AH. Exact solutions of thin film flows. *Nonlinear Dynamics* 2007; **50**:229–233.
19. Hayat T, Mahomed FM. Note on an exact solution for the pipe flow of a third grade. *Acta Mechanica* 2007; **190**:249–263.
20. Hayat T. Exact solutions to rotating flows of a Burgers' fluid. *Computers and Mathematics with Applications* 2006; **52**:1413–1424.
21. Hayat T, Ellahi R, Mahomed FM. Exact solutions for thin film flow of a third grade fluid down an inclined plane. *Chaos*, *Solitons and Fractals* 2008; **38**:1336–1341.
22. Liao SJ. *Beyond Perturbation*: *Introduction to the Homotopy Analysis Method*. Chapman & Hall, CRC Press: Boca Raton, FL, 2003.
23. Liao SJ. Notes on the homotopy analysis: some definitions and theorems. *Communications in Nonlinear Science and Numerical Simulation* 2008; DOI: 10.1016/j.cnsns.2008.04.013.
24. Hayat T, Khan M, Ayub M. On non-linear flows with slip boundary condition. *Zeitschrift für Angewandte Mathematik und Physik* 2005; **56**:1012–1029.
25. Sajid M, Hayat T, Asghar S. Comparison between the HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt. *Nonlinear Dynamics* 2007; **50**:27–35.
26. Abbasbandy S. Soliton solutions for the Fitzhugh–Nagumo equation with the homotopy analysis method. *Applied Mathematical Modelling* 2008; **32**:2706–2714.
27. Abbasbandy S, Samadian Zakaria F. Soliton solutions for the fifth-order KdV equation with the homotopy analysis method. *Nonlinear Dynamics* 2008; **51**:83–87.
28. Abbasbandy S, Yürüsoy M, Pakdemirli M. The analysis approach of boundary layer equations of power-law fluids of second grade. *Zeitschrift für Naturforschung A* 2008; **63**(a):564–570.
29. Bataineh AS, Noorani MSM, Hashim I. Approximate solutions of singular two-point BVPs by modified homotopy analysis method. *Physics Letters A* 2008; **372**:4062–4066.
30. Cheng J, Liao SJ, Mohapatra RN, Vajravelu K. Series solutions of nano boundary layer flows by means of the homotopy analysis method. *Journal of Mathematical Analysis and Applications* 2008; **343**:233–245.
31. Sajid M, Hayat T. Comparison of HAM and HPM methods in nonlinear heat conduction and convection equations. *Nonlinear Analysis*: *Real World Applications* 2008; **9**:2296–2301.
32. Van Gorder RA, Vajravelu K. Analytic and numerical solutions to the Lane–Emden equation. *Physics Letters A* 2008; **372**:6060–6065.
33. Xu H, Liao SJ, Pop I. Series solutions of unsteady free convection flow in the stagnation-point region of a three-dimensional body. *International Journal of Thermal Sciences* 2008; **47**:600–608.
34. Liao SJ. A general approach to get series solution of non-similarity boundary-layer flows. *Communications in Nonlinear Science and Numerical Simulation* 2008; DOI: 10.1016/j.cnsns.2008.06.013.

DOI: 10.1002/fld